

Stochastic Calculus with respect to G -Brownian Motion Viewed through Rough PathsPeng, ShiGe^{1,2,*} & Zhang, HuiLin^{1,2}¹*School of Mathematics, Shandong University, Jinan 250100, China;*²*Institute for Advanced Research, Shandong University, Jinan 250100, China;**Email: peng@sdu.edu.cn, huilinzhang2014@gmail.com*

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Abstract In this paper, we study rough path properties of stochastic integrals of Itô's type and Stratonovich's type with respect to G -Brownian motion. The roughness of G -Brownian Motion is estimated and then the pathwise Norris lemma in G -framework is obtained.

Keywords rough paths, roughness of G -Brownian motion, Norris lemma.

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1 Introduction

Since Pardoux and Peng [18], backward stochastic differential equations (BSDEs) receive much attention and are widely applied in many areas such as stochastic control, financial mathematics, PDEs (see [13], [16], [19], [20], [21] for example). However, BSDEs fail to give a probabilistic explanation to fully nonlinear PDEs. Motivated by such disadvantages of BSDEs and applications in financial mathematics, G -expectation theory was introduced by Peng in [22], [23]. G -expectation is a time consistent sublinear expectation, which is obtained from a fully nonlinear parabolic PDE, called G -heat equation, with the canonical process B_t as G -Brownian motion. Stochastic analysis and the corresponding BSDEs in G -framework are established in [22], [23], [24], [10], [11].

Rough path theory was introduced by Lyons in his pioneer work [14], to give a well defined integration when the driving path is not smooth (with p -variation for $p \geq 2$). The universal limit theorem for differential equations driven by rough paths was obtained and the continuity of Itô-Lyons map for the corresponding rough differential equations (RDEs for short) was firstly established by Lyons. Later, Gubinelli expanded integrands of rough integral from one-forms to controlled paths (see [5], [6]). Geng et al first investigated rough path properties of G -Brownian motion in [7]. Firstly, G -Brownian motion is lifted as geometric rough paths. Then, some basic relations between SDEs and RDEs driven by G -Brownian motion were established. These results allowed them to prove the existence and uniqueness theorem of SDEs driven by G -Brownian motion on differentiable manifolds.

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A natural question is what is the relation between rough integrals and stochastic integrals with respect to G -Brownian motion. Furthermore, does the G -Brownian motion possesses the roughness pathwisely? In this paper we study the rough path properties based on the α -Hölder continuity of G -Brownian motion, of which the enhancement could be completed by a generalized Kolmogorov's criterion for rough paths under G -expectation framework, which is more direct and probabilistic compared with [7]. Moreover, the cross variation of Itô's process under G -Brownian motion framework is studied, through which the Stratonovich integral is defined. Then, the relation among rough integral, Itô integral and Stratonovich integral with respect to G -Brownian motion is established. At last, the roughness of G -Brownian motion is calculated and then the Norris lemma for stochastic integral with respect to G -Brownian motion is obtained. Further work about applications in finance such as no arbitrage hedging and superhedging could be possibly available in later papers by authors.

The paper is organized as follows. In Section 2, we recall some basic definitions and results in G -expectation theory and rough path theory. Then in Section 3, G -Brownian motion is lifted as rough paths, and Itô integral with respect to G -Brownian motion is proved to be equivalent to the corresponding rough integral. Then, the quadratic variation of G -Itô process is introduced and the Stratonovich integral with respect to G -Brownian motion is defined. Similarly, the equivalence between G -Stratonovich integral and the corresponding rough integral is established. In Section 4, the θ -Hölder roughness of G -Brownian motion is studied, and then the pathwise Norris lemma in G -framework is obtained.

2 Preliminaries about the G -expectation and Rough Paths

In this part, we give some definitions and results of G -expectation and rough path theories. The proofs can be found in [3], [15], [22], [24].

2.1 The rough path theory

For rough path theory presented in this paper, we adopt the framework of Friz and Hairer [3], see also Gubinelli [5].

Denote by $\mathbb{R}^m \otimes \mathbb{R}^n$ the algebraic tensor of two Euclidean spaces. For any path on some interval $[0, T]$ with values in a \mathbb{R}^d , its α -Hölder norm(semi-norm) is defined by

$$\|X\|_\alpha = \sup_{0 \leq s < t \leq T} \frac{|X_{s,t}|}{|t-s|^\alpha},$$

where $X_{s,t} = X_t - X_s$, for any path X .

Denote $\mathcal{C}^\alpha([0, T], \mathbb{R}^d)$ as the space of paths with finite α -Hölder norm and values in \mathbb{R}^d . Similarly, a mapping \mathbb{X} from $[0, T]^2$ to $\mathbb{R}^d \otimes \mathbb{R}^d$ is attached with norm

$$\|\mathbb{X}\|_{2\alpha} = \sup_{0 \leq s \neq t \leq T} \frac{|\mathbb{X}_{s,t}|}{|t-s|^{2\alpha}},$$

whenever it's finite.

A rough path on some interval $[0, T]$ with values in \mathbb{R}^d includes a "rough" continuous path $X : [0, T] \rightarrow \mathbb{R}^d$, along with its "iterated integration" part $\mathbb{X} : [0, T]^2 \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$, which satisfies "Chen's identity",

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = X_{s,u} \otimes X_{u,t}, \quad (2.1)$$

and Hölder continuity.

In the sequel, suppose $\alpha \in (\frac{1}{3}, \frac{1}{2})$ for the need of rough integral with respect to G -Brownian motion.

Definition 2.1. For a fixed α , the space of rough paths $\mathcal{C}^\alpha([0, T], \mathbb{R}^d)$ on $[0, T]$ consists of pairs (X, \mathbb{X}) satisfying “Chen’s identity” (2.1) and the condition of finite α -Hölder norm and 2α -Hölder norm respectively for X and \mathbb{X} . For any $\mathbf{X} := (X, \mathbb{X}) \in \mathcal{C}^\alpha([0, T], \mathbb{R}^d)$, define its semi-norm as the following

$$\|\mathbf{X}\|_{\mathcal{C}^\alpha} := \|X\|_\alpha + (\|\mathbb{X}\|_{2\alpha})^{\frac{1}{2}}.$$

Definition 2.2. A path $Y \in \mathcal{C}^\alpha([0, T], \mathbb{R}^m)$ is said to be controlled by a given path $X \in \mathcal{C}^\alpha([0, T], \mathbb{R}^d)$, if there exists $Y' \in \mathcal{C}^\alpha([0, T], \mathcal{L}(\mathbb{R}^d, \mathbb{R}^m))$, such that the remainder term

$$R_{s,t}^Y := Y_{s,t} - Y'_s X_{s,t},$$

satisfies $\|R^Y\|_{2\alpha} < \infty$.

Here $\mathcal{L}(\mathbb{R}^d, \mathbb{R}^m)$ means the space of linear functions from \mathbb{R}^d to \mathbb{R}^m , which is indeed \mathbb{R}^{dm} . Denote the collection of controlled rough paths by $\mathcal{D}_X^{2\alpha}([0, T], \mathbb{R}^m)$. In addition, Y' is called the *Gubinelli derivative* of Y . For $(Y, Y') \in \mathcal{D}_X^{2\alpha}([0, T], \mathbb{R}^m)$, we define its semi-norm by $\|Y, Y'\|_{X, 2\alpha} := \|Y'\|_\alpha + \|R^Y\|_{2\alpha}$.

For example, given any $F \in \mathcal{C}_b^2(\mathbb{R}^d, \mathbb{R}^m)$, the set of bounded functions from \mathbb{R}^d to \mathbb{R}^m with bounded derivatives up to order 2, one can easily check that $(Y, Y') := (F(X), DF(X)) \in \mathcal{D}_X^{2\alpha}([0, T], \mathbb{R}^m)$. In general, Y' is not uniquely determined by Y , especially when X is rather smooth. However, if the underlying path X is truly rough, Y' can be uniquely decided by Y (see [4], [3] for details).

The next theorem for the definition of rough integral based on controlled paths is obtained in [5], also see [3], [14], [15].

Theorem 2.3. (Gubinelli, Lyons) Suppose $\mathbf{X} \in \mathcal{C}^\alpha([0, T], \mathbb{R}^d)$, and $(Y, Y') \in \mathcal{D}_X^{2\alpha}([0, T], \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n))$. Then the following compensated Riemann sum converges.

$$\int_0^T Y d\mathbf{X} := \lim_{|\mathcal{P}| \rightarrow 0} \sum_{(s,t) \in \mathcal{P}} (Y_s X_{s,t} + Y'_s \mathbb{X}_{s,t}), \quad (2.2)$$

where \mathcal{P} are partitions of $[0, T]$, with modulus $|\mathcal{P}| \rightarrow 0$. Furthermore, one has the bound

$$\left| \int_s^t Y_r d\mathbf{X}_r - Y_s X_{s,t} - Y'_s \mathbb{X}_{s,t} \right| \leq K (\|X\|_\alpha \|R^Y\|_{2\alpha} + \|\mathbb{X}\|_{2\alpha} \|Y'\|_\alpha) |t - s|^{3\alpha}, \quad (2.3)$$

where K is a constant depending only on α .

Here one should note that $\mathcal{L}(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n)) \cong \mathcal{L}(\mathbb{R}^d \otimes \mathbb{R}^d, \mathbb{R}^n)$, so $\int_0^T Y d\mathbf{X} \in \mathbb{R}^n$, and $m = dn$ where m is in the definition of controlled paths.

The Norris lemma was first established in [17], and is viewed as a quantitative version of Doob-Meyer’s decomposition. A deterministic quantitative Norris Lemma is given in [1]. It means that a rough integral can be distinguished from a rather “smooth” integral, essentially by the uniqueness of Gubinelli’s derivative, when the given rough path is “truly rough”. Precisely, one has the following definition and theorem.

Definition 2.4. A path $X \in \mathcal{C}^\alpha([0, T], \mathbb{R}^d)$ is said to be θ -Hölder rough for some given $\theta \in (0, 1)$, on the scale of $\varepsilon_0 > 0$, if there exists a constant $L > 0$, such that for any $a \in \mathbb{R}^d$, $s \in [0, T]$, and $\varepsilon \in (0, \varepsilon_0]$, there always exists $t \in [0, T]$, satisfying

$$|t - s| < \varepsilon, \text{ and } |a \cdot X_{s,t}| \geq L\varepsilon^\theta |a|.$$

The largest value of such L is called the modulus of θ -Hölder roughness of X , denoted by $L_\theta(X)$. It is obvious that the modulus $L_\theta(X)$ has the following expression:

$$L_\theta(X) = \inf_{|a|=1, s \in [0, T], \varepsilon \in (0, \varepsilon_0]} \sup_{|t-s| \leq \varepsilon} \frac{1}{\varepsilon^\theta} |a \cdot X_{s,t}|. \quad (2.4)$$

Theorem 2.5. (Norris lemma for rough paths) Suppose $\mathbf{X} = (X, \mathbb{X}) \in \mathcal{C}^\alpha([0, T], \mathbb{R}^d)$, with X θ -Hölder rough for some $\theta < 2\alpha$. Given $(Y, Y') \in \mathcal{D}_X^{2\alpha}([0, T], \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n))$ and $Z \in \mathcal{C}^\alpha([0, T], \mathbb{R}^n)$, set

$$I_t = \int_0^t Y_s d\mathbf{X}_s + \int_0^t Z_s ds, \quad (2.5)$$

and

$$\mathcal{R} = 1 + L_\theta(X)^{-1} + \|\mathbf{X}\|_{\mathcal{C}^\alpha} + \|Y, Y'\|_{X, 2\alpha} + |Y_0| + |Y'_0| + \|Z\|_\alpha + |Z_0|. \quad (2.6)$$

Then one has the bound

$$\|Y\|_\infty + \|Z\|_\infty \leq M\mathcal{R}^q \|I\|_\infty^r, \quad (2.7)$$

for some constant M, q and r , only depending on α, θ, T .

2.2 The G -expectation theory

To introduce G -expectation theory, firstly we need to give a short description of the sublinear expectation. Let Ω be a given set and \mathcal{H} be a linear space of real valued functions on Ω containing constants. Furthermore, suppose $\varphi(X_1, \dots, X_n) \in \mathcal{H}$ if $X_1, \dots, X_n \in \mathcal{H}$ for $\varphi \in \mathcal{C}_{b.Lip}(\mathbb{R}^n)$, the space of bounded Lipschitz functions. The space Ω is the sample space and \mathcal{H} is the space of random variables.

Definition 2.6. A sublinear expectation $\hat{\mathbb{E}}$ is a functional $\hat{\mathbb{E}} : \mathcal{H} \rightarrow \mathbb{R}$ satisfying:

- $\hat{\mathbb{E}}[c] = c, \quad \forall c \in \mathbb{R};$
- $\hat{\mathbb{E}}[X_1] \geq \hat{\mathbb{E}}[X_2] \quad \text{if } X_1 \geq X_2;$
- $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X], \quad \lambda \geq 0 \quad X \in \mathcal{H};$
- $\hat{\mathbb{E}}[X + Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y], \quad X, Y \in \mathcal{H}.$

The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sublinear expectation space.

Definition 2.7. In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, a random vector $Y = (Y_1, \dots, Y_n)$, $Y_i \in \mathcal{H}$, $i = 1 \dots n$, is said to be independent of another random vector $X = (X_1, \dots, X_m)$, $X_i \in \mathcal{H}$ under $\hat{\mathbb{E}}[\cdot]$, if for every test function $\varphi \in \mathcal{C}_{b.Lip}(\mathbb{R}^m \times \mathbb{R}^n)$, we have $\hat{\mathbb{E}}[\varphi(X, Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x, Y)]_{x=X}]$.

Remark 2.8. If Y is independent of X , one fails to get that X is independent of Y automatically. Indeed, this is a main difference between G -expectation theory and the classical case. There are nontrivial examples explaining this point. See Chapter 1 in [24].

Definition 2.9. Let X_1 and X_2 be two n -dimensional random vectors defined respectively in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$. They are called identically distributed, denoted by $X_1 \stackrel{d}{=} X_2$, if $\hat{\mathbb{E}}_1[\varphi(X_1)] = \hat{\mathbb{E}}_2[\varphi(X_2)]$, for all $\varphi \in \mathcal{C}_{b.Lip}(\mathbb{R}^n)$.

Definition 2.10. (G -normal distribution) A d -dimensional random vector $X = (X_1, \dots, X_d)$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called G -normally distributed if $\hat{\mathbb{E}}[|X|^3] < \infty$ and for each $a, b \geq 0$

$$aX + b\bar{X} \stackrel{d}{=} \sqrt{a^2 + b^2} X,$$

where \bar{X} is an independent copy of X , i.e., $\bar{X} \stackrel{d}{=} X$, \bar{X} independent of X , and

$$G(A) := \frac{1}{2} \hat{\mathbb{E}}[X'AX] : \mathbb{S}_d \rightarrow \mathbb{R},$$

Here \mathbb{S}_d denotes the collection of $d \times d$ symmetric matrices.

By Theorem 1.6 in Chapter 3 of [24], we know that if $X = (X_1, \dots, X_d)$ is G -normally distributed, $u(t, x) := \hat{\mathbb{E}}[\varphi(x + \sqrt{t}X)]$, $(t, x) \in [0, \infty) \times \mathbb{R}^d$, is the unique viscosity solution of the following G -heat equation:

$$\partial_t u - G(D_x^2 u) = 0, \quad u(0, x) = \varphi(x), \quad (2.8)$$

with function G defined as above.

Conversely, fixed any monotonic, sublinear function $G(\cdot) : \mathbb{S}_d \rightarrow \mathbb{R}$, one could construct the sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$.

Now let $\Omega = \mathcal{C}^0(\mathbb{R}^+, \mathbb{R}^d)$, the space of \mathbb{R}^d valued continuous paths $(\omega)_{t \geq 0}$ vanishing at the origin. Denote the coordinate process by B_t and $u^{\varphi(\cdot)}(t, x)$ the unique viscosity solution to the G -heat equation (2.8) with initial function φ . Define $L_{ip}(\Omega_T) := \{\varphi(B_{t_1 \wedge T}, \dots, B_{t_k \wedge T}) : k \in \mathbb{N}, t_1, \dots, t_k \in [0, \infty), \varphi \in \mathcal{C}_{b, Lip}(\mathbb{R}^{k \times d})\}$ for any $T > 0$ and $L_{ip}(\Omega) := \bigcup_{n=1}^{\infty} L_{ip}(\Omega_n)$. We define a mapping $\hat{\mathbb{E}}$ from $L_{ip}(\Omega)$ to \mathbb{R} by recursively solving the G -heat equation:

$$\hat{\mathbb{E}}[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}})] := \hat{\mathbb{E}}[\varphi^{t_n - t_{n-1}}(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_{n-1}} - B_{t_{n-2}})], \quad (2.9)$$

where $\varphi^{t_n - t_{n-1}}(x_1, \dots, x_{n-1}) := u^{\varphi(x_1, \dots, x_{n-1}, \cdot)}(t_n - t_{n-1}, 0)$. One can check that $\hat{\mathbb{E}}[\cdot]$ is well defined and it is a sublinear expectation on $L_{ip}(\Omega)$. Furthermore, one could define the time consistent conditional expectation $\hat{\mathbb{E}}[\cdot | \Omega_s]$ as the mapping from $L_{ip}(\Omega)$ to $L_{ip}(\Omega_s)$ by

$$\hat{\mathbb{E}}[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}}) | \Omega_s] = \psi(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_s - B_{t_{i-1}}), \quad s \in [t_{i-1}, t_i], \quad (2.10)$$

with $\psi(x_1, \dots, x_i) = \hat{\mathbb{E}}[\varphi(x_1, \dots, x_i + B_{t_i} - B_s, \dots, B_{t_n} - B_{t_{n-1}})]$. Here is a collection of properties for this mapping.

- $\hat{\mathbb{E}}[\xi | \Omega_t] = \xi$, for any $\xi \in L_{ip}(\Omega_t)$.
- $\hat{\mathbb{E}}[X + Y | \Omega_t] \leq \hat{\mathbb{E}}[X | \Omega_t] + \hat{\mathbb{E}}[Y | \Omega_t]$.
- $\hat{\mathbb{E}}[\xi X | \Omega_t] = \xi^+ \hat{\mathbb{E}}[X | \Omega_t] + \xi^- \hat{\mathbb{E}}[-X | \Omega_t]$, for any $\xi \in L_{ip}(\Omega_t)$
- $\hat{\mathbb{E}}[\hat{\mathbb{E}}[X | \Omega_t] | \Omega_s] = \hat{\mathbb{E}}[X | \Omega_{t \wedge s}]$, in particular, $\hat{\mathbb{E}}[\hat{\mathbb{E}}[X | \Omega_t]] = \hat{\mathbb{E}}[X]$.
- $\hat{\mathbb{E}}[X | \Omega_t] = \hat{\mathbb{E}}[X]$, if X is independent of $L_{ip}(\Omega_t)$.
- $\hat{\mathbb{E}}[X + \xi | \Omega_t] = \hat{\mathbb{E}}[X | \Omega_t] + \xi$, for any $\xi \in L_{ip}(\Omega_t)$, $X \in L_{ip}(\Omega)$.

From now on, we suppose the function G non-degenerate, i.e., there exists two constants $0 < \underline{\sigma}^2 \leq \bar{\sigma}^2 < \infty$, such that

$$\frac{1}{2} \underline{\sigma}^2 \text{tr}(A - B) \leq G(A) - G(B) \leq \frac{1}{2} \bar{\sigma}^2 \text{tr}(A - B).$$

In the case that $\bar{\sigma} = \underline{\sigma}$, the function G is linear, so G -framework is the classical Wiener case.

For each $p \geq 1$, $L_G^p(\Omega_T)$ denotes the completion of the linear space $L_{ip}(\Omega_T)$, under norm $\|\cdot\|_{L_G^p} := \{\hat{\mathbb{E}}[\|\cdot\|^p]\}^{\frac{1}{p}}$. Obviously, for any $p \leq q$, $L_G^p \subseteq L_G^q$. Furthermore, the conditional expectation $\hat{\mathbb{E}}[\cdot | \Omega_t]$ could be continuously extended to a mapping from $L_G^1(\Omega)$ to $L_G^1(\Omega_t)$ and the extended mapping adopts the above properties.

To give a description of elements in L_G^p , Denis, Hu and Peng gave the following representation of $\hat{\mathbb{E}}[\cdot]$ by stochastic control methods in [2]. Also see Hu and Peng [12] for an intrinsic and probabilistic method.

Theorem 2.11. Assume Γ is a bounded, convex and closed subset of $\mathbb{R}^{d \times d}$, which represents function G , i.e.,

$$G(A) = \frac{1}{2} \sup_{\gamma \in \Gamma} \text{tr}(A \gamma \gamma'), \text{ for } A \in \mathbb{S}_d.$$

Denote the Wiener measure by P^0 . Then, for any time sequence $0 = t_0 < t_1 < \dots < t_k$, the G -expectation has the following representation

$$\begin{aligned}\hat{\mathbb{E}}[\varphi(B_{t_0,t_1}, \dots, B_{t_{k-1},t_k})] &= \sup_{a \in \mathcal{A}^\Gamma} E_{P^0}[\varphi(\int_0^{t_1} a_s dB_s, \dots, \int_{t_{k-1}}^{t_k} a_s dB_s)] \\ &= \sup_{P^a \in \mathcal{P}^\Gamma} E_{P^a}[\varphi(B_{t_0,t_1}, \dots, B_{t_{k-1},t_k})],\end{aligned}$$

where \mathcal{A}^Γ is the set of progressively measurable processes with values in Γ and \mathcal{P}^Γ is the set of laws of $\int_0^\cdot a_s dB_s$ with $a \in \mathcal{A}^\Gamma$ under Wiener measure. Furthermore, \mathcal{P}^Γ is tight.

According to this theorem, one could extend $\hat{\mathbb{E}}$ from L_G^p to any Borel measurable random variable by defining

$$\|\cdot\|_{\mathbb{L}^p} := \sup_{P^a \in \mathcal{P}^\Gamma} E_{P^a}^{\frac{1}{p}}[\|\cdot\|^p].$$

It is simple to check that if $X \in L_G^p$, then $\|X\|_{\mathbb{L}^p} = \|X\|_{L_G^p}$.

Next, we introduce the capacity corresponding to the G -expectation and give the description of L_G^p . Define

$$\hat{c}(A) := \sup_{P \in \mathcal{P}^\Gamma} P(A), \text{ for } A \in \mathcal{B}(\Omega_T).$$

Definition 2.12. A property is said to hold “quasi-surely” (q.s.) with respect to \hat{c} , if it holds true outside a \hat{c} -polar set (Borel set with capacity 0), and is denoted by \hat{c} -q.s..

Definition 2.13. A process Y on $[0, T]$ is said to be a quasi-surely modification of another process X , if for any $t \in [0, T]$

$$Y_t = X_t, \quad \hat{c}\text{-q.s..}$$

If a property stands true \hat{c} -q.s., then for any $P \in \mathcal{P}^\Gamma$, it holds true P -a.s.. By the definition of L_G^p , we do not distinguish two random variables if they are equal outside a polar set.

Definition 2.14. Equip the space Ω_T with the uniform topology. A mapping X on Ω_T with values in \mathbb{R} is said to be quasi-continuous if for any $\varepsilon > 0$, there exists an open set O , with $\hat{c}(O) < \varepsilon$ such that X is continuous in O^c .

Definition 2.15. One says that $X : \Omega_T \rightarrow \mathbb{R}$ has a quasi-continuous version if there exists a quasi-continuous function Y , such that $X = Y$, \hat{c} -q.s..

Theorem 2.16. One has the following representation for L_G^1 ,

$$L_G^1(\Omega_T) = \{X \in \mathcal{B}(\Omega_T) : X \text{ has a quasi-continuous version, } \lim_{n \rightarrow \infty} \|X|1_{\{|X|>n\}}\|_{\mathbb{L}^1} = 0\}.$$

Proposition 2.17. Assume that $(X_n)_{n \geq 1}$ is a sequence of random variables, and converges to X in the sense of $\|\cdot\|_{\mathbb{L}^p}$. Then the convergence holds in the sense of capacity, i.e., for any $\varepsilon > 0$,

$$\hat{c}(|X_n - X| > \varepsilon) \xrightarrow{n} 0.$$

Furthermore, there exists a subsequence $(X_{n_k})_{k \geq 1}$ converging to X quasi-surely.

Remark 2.18. It is vital to point out that though the above proposition holds true in the G -framework, even sup linear expectation framework, the dominated convergence theorem (the quasi-surely version), and the claim that quasi-surely convergence implies convergence in capacity, all fail in G -framework.

Now we introduce the stochastic integral (Itô's integral) for one-dimension case in G -framework.

Denote $M_G^{p,0}(0, T)$ the collection of processes with form

$$\eta_t(\omega) = \sum_{i=0}^{N-1} \xi_i(\omega) 1_{[t_i, t_{i+1})}(t),$$

for a partition $\{0 = t_0 < \dots < t_N = T\}$ and $\xi_i \in L_{ip}(\Omega_{t_i}), i = 0 \dots N-1$. Then denote by $M_G^p(0, T)$ the completion of $M_G^{p,0}(0, T)$ under norm $\|\cdot\|_{M_G^p} := \{\hat{\mathbb{E}} \int_0^T |\eta_s|^p ds\}^{\frac{1}{p}}$.

Definition 2.19. For each $\eta \in M_G^{2,0}(0, T)$, one has the mapping I from $M_G^{2,0}(0, T)$ to $L_G^2(\Omega_T)$:

$$I(\eta) = \int_0^T \eta_s dB_s := \sum_{i=0}^{N-1} \xi_i (B_{t_{i+1}} - B_{t_i}). \quad (2.11)$$

It has been shown (see [22], [23], [24]) that the mapping is continuous and can be extended to the completion space $M_G^2(0, T)$. Define the quadratic variation processes $\langle B \rangle$ of G -Brownian motion by

$$\langle B \rangle_t := B_t^2 - 2 \int_0^t B_s dB_s. \quad (2.12)$$

It can be shown that $\underline{\sigma}^2 \leq \frac{d\langle B \rangle_t}{dt} \leq \bar{\sigma}^2$, $\hat{c}-q.s.$, where $\underline{\sigma} = \sqrt{-\hat{\mathbb{E}}[-B_1^2]}$ and $\bar{\sigma} = \sqrt{\hat{\mathbb{E}}[B_1^2]}$. In G -expectation theory, $\langle B \rangle$ shares properties of independent stationary increment just as G -Brownian motion. Moreover, the following integral of a process in $M_G^{1,0}(0, T)$ can be continuously extended to the completion $M_G^1(0, T)$.

$$\int_0^T \eta_t d\langle B \rangle_t := \sum_{i=0}^{N-1} \xi_i (\langle B \rangle_{t_{i+1}} - \langle B \rangle_{t_i}) : M_G^{1,0}(0, T) \rightarrow L_G^1(\Omega_T), \quad (2.13)$$

where η is defined as above, only L_G^2 replaced by L_G^1 .

For the multi-dimensional case, one could obtain similar results. Indeed, let $(B_t)_{t \geq 0}$ be a d -dimensional G -Brownian motion. For any $a \in \mathbb{R}^d$, $B^a := a \cdot B$ is still a G_a -Brownian motion. Then according to results in one-dimensional case, one could define integrals with respect to $B^a, \langle B^a \rangle$, and obtain continuity for these mappings. Furthermore, the mutual variation process $\langle B^a, B^{\bar{a}} \rangle_t$ could be defined by polarization.

At last, we end this subsection with Itô's formula in G -framework. The proof could also be obtained in [24].

Theorem 2.20. Let Φ be a twice continuously differentiable function on \mathbb{R}^n with polynomial growth for the first and second order derivatives. Suppose X is a Itô process, i.e.

$$X_t^\nu = X_0^\nu + \int_0^t \alpha_s^\nu ds + \int_0^t \eta_s^{\nu ij} d\langle B^i, B^j \rangle + \int_0^t \beta_s^{\nu j} dB_s^j$$

where $\nu = 1, \dots, n$, $i, j = 1, \dots, d$, $\alpha_s^\nu, \eta_s^{\nu ij}, \beta_s^{\nu j}$ are bounded processes in $M_G^2(0, T)$. Here and from now on, repeated indices means summation over the same ones. Then for each $t \geq s \geq 0$ we have in $L_G^2(\Omega_t)$:

$$\begin{aligned} \Phi(X_t) - \Phi(X_s) &= \int_s^t \partial_{x^\nu} \Phi(X_u) \beta_u^{\nu j} dB_u^j + \int_s^t \partial_{x^\nu} \Phi(X_u) \alpha_u^\nu du \\ &+ \int_s^t [\partial_{x^\nu} \Phi(X_u) \eta_u^{\nu ij} + \frac{1}{2} \partial_{x^\mu x^\nu}^2 \Phi(X_u) \beta_u^{\mu i} \beta_u^{\nu j}] d\langle B^i, B^j \rangle_u. \end{aligned}$$

3 G -Stochastic Integral as Rough Integral

Firstly we give the G -expectation version of Kolmogorov criterion for rough paths, the proof of which is adapted from the classical case (see Theorem 3.1 in [3]).

Theorem 3.1. For fixed $q \geq 2, \beta > \frac{1}{q}$, assume $X(\omega) : [0, T] \rightarrow \mathbb{R}^d$ and $\mathbb{X}(\omega) : [0, T]^2 \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ are processes with $X_t \in L_G^q(\Omega_T), \mathbb{X}_{s,t} \in L_G^{\frac{q}{2}}(\Omega_T), \forall s, t \in [0, T]$, and satisfy relation (2.1), $\hat{c} - q.s.$. If for any $s, t \in [0, T]$, one has bounds

$$\|X_{s,t}\|_{L_G^q} \leq C|t-s|^\beta, \quad \|\mathbb{X}_{s,t}\|_{L_G^{\frac{q}{2}}} \leq C|t-s|^{2\beta}, \quad (3.1)$$

for some constant C . Then for all $\alpha \in [0, \beta - \frac{1}{q})$, (X, \mathbb{X}) has a $\hat{c} - q.s.$ continuous modification, and there exist $K_\alpha \in L_G^q, \mathbb{K}_\alpha \in L_G^{\frac{q}{2}}$ such that for any $s, t \in [0, T]$, one has inequalities

$$|X_{s,t}| \leq K_\alpha |t-s|^\alpha, \quad |\mathbb{X}_{s,t}| \leq \mathbb{K}_\alpha |t-s|^{2\alpha}, \quad \hat{c} - q.s.. \quad (3.2)$$

In particular, if $\beta - \frac{1}{q} > \frac{1}{3}$, then $\hat{c} - q.s.$ $\mathbf{X} = (X, \mathbb{X})$ belongs to $\mathcal{C}^\alpha([0, T], \mathbb{R}^d)$, for any $\alpha \in (\frac{1}{3}, \beta - \frac{1}{q})$.

Proof. Without loss of generality suppose $T=1$, and define dyadic partition as $D_n = \{i2^{-n}, i = 0 \dots 2^n\}$. Set

$$K_n = \sup_{t \in D_n} |X_{t, t+2^{-n}}|, \quad \mathbb{K}_n = \sup_{t \in D_n} |\mathbb{X}_{t, t+2^{-n}}|.$$

Note that since D_n are finite sets, K_n, \mathbb{K}_n belong to L_G^q and $L_G^{\frac{q}{2}}$ respectively. Furthermore, one has bounds

$$\begin{aligned} \hat{\mathbb{E}}(K_n^q) &\leq \sum_{D_n} \hat{\mathbb{E}}|X_{t, t+2^{-n}}|^q \leq C^q \left(\frac{1}{2^n}\right)^{\beta q - 1} \\ \hat{\mathbb{E}}(\mathbb{K}_n^{\frac{q}{2}}) &\leq \sum_{D_n} \hat{\mathbb{E}}|\mathbb{X}_{t, t+2^{-n}}|^{\frac{q}{2}} \leq C^{\frac{q}{2}} \left(\frac{1}{2^n}\right)^{\beta q - 1} \end{aligned}$$

For any $s, t \in \bigcup_n D_n$, there exists m such that $2^{-m-1} < t-s \leq 2^{-m}$, and a partition, $s = \tau_0 < \tau_1 < \dots < \tau_N = t$, with $(\tau_i, \tau_{i+1}) \in D_k$, for some $k \geq m+1$. Also, we can choose such a partition that at most two intervals in this partition are taken from the same D_k for any fixed $k \geq m+1$.

Then one obtains

$$|X_{s,t}| \leq \max_{0 \leq i \leq N} |X_{s, \tau_i}| \leq \sum_{i=0}^{N-1} |X_{\tau_i, \tau_{i+1}}| \leq 2 \sum_{n \geq m+1} K_n.$$

It follows that

$$\frac{|X_{s,t}|}{|t-s|^\alpha} \leq 2 \sum_{n \geq m+1} (2^n)^\alpha K_n \leq K_\alpha,$$

where $K_\alpha := 2 \sum_{n \geq 0} 2^{n\alpha} K_n$. We can easily check that $K_\alpha \in L_G^q$ since $K_n \in L_G^q$.

For the second order part \mathbb{X} , by “Chen’s identity”, one has the following inequalities,

$$\begin{aligned} |\mathbb{X}_{s,t}| &= \left| \sum_{i=0}^{N-1} (\mathbb{X}_{\tau_i, \tau_{i+1}} + X_{s, \tau_i} \otimes X_{\tau_i, \tau_{i+1}}) \right| \\ &\leq 2 \sum_{n \geq m+1} \mathbb{K}_n + \max_{0 \leq i \leq N} |X_{s, \tau_{i+1}}| \sum_{j=0}^{N-1} |X_{\tau_j, \tau_{j+1}}| \\ &\leq 2 \sum_{n \geq m+1} \mathbb{K}_n + (2 \sum_{n \geq m+1} K_n)^2. \end{aligned}$$

Then one obtains

$$\frac{|\mathbb{X}_{s,t}|}{|t-s|^{2\alpha}} \leq \sum_{n \geq 1} 2\mathbb{K}_n 2^{2n\alpha} + K_\alpha^2,$$

the right side of which can be checked to belong to $L_G^{\frac{q}{2}}$. \square

3.1 G -Itô integral as rough integral

Let us consider the G -Brownian motion as rough paths. Suppose $B = (B^{(1)}, \dots, B^{(d)})$ is a d -dimensional G -Brownian motion and

$$\hat{\mathbb{E}}[|B^{(i)} B^{(j)}|] = \bar{\sigma}_{ij}^2, \quad -\hat{\mathbb{E}}[-|B^{(i)} B^{(j)}|] = \underline{\sigma}_{ij}^2.$$

For simplicity, suppose for some positive number $\bar{\sigma}$, $\underline{\sigma}_{ij} \leq \bar{\sigma}_{ij} < \bar{\sigma}$ for any i, j . Firstly, it is obvious that the lifted G -Brownian motion, $(B, \mathbb{B}) := (B, \int_s^t B_{s,r} dB_r) = (\{B^{(i)}\}_{i=1}^d, \{\int_s^t B_{s,r}^{(i)} dB_r^{(j)}\}_{1 \leq i, j \leq d})$ satisfies (2.1). There remains the analytic condition to be checked. With an application of Theorem 3.1, the following proposition would stand for our claim that the lifted G -Brownian motion belongs to the rough path space \mathcal{C}^α , $\hat{c} - q.s.$.

Proposition 3.2. One has the following inequalities

$$\|B_{s,t}\|_{L_G^q} \leq C_{q,\bar{\sigma}} |t-s|^{\frac{1}{2}}, \quad \|\mathbb{B}_{s,t}\|_{L_G^{\frac{q}{2}}} \leq C_{q,\bar{\sigma}} |t-s|, \quad \text{for any } q \geq 2,$$

where $C_{q,\bar{\sigma}}$ is a constant depending on k and $\bar{\sigma}$.

Proof. It is obvious that $\|B_{s,t}\|_{L_G^q} \leq C_q |t-s|^{\frac{1}{2}}$. Thanks to the property of stationary and independent increment for G -Brownian motion, only $\hat{\mathbb{E}}|\int_0^t B_r dB_r|^{2k} \leq C_k t^{2k}$, for any $k \geq 1$, left to be checked.

Note that $\int_0^t B_r^{(i)} dB_r^{(j)}$ is a square integrable continuous martingale under each $P \in \mathcal{P}^\Gamma$ by Theorem 2.11. A combination of Burkholder-Davis-Gundy inequality and Jensen's inequality tells that

$$\begin{aligned} \hat{\mathbb{E}}|\int_0^t B_r^{(i)} dB_r^{(j)}|^{2k} &\leq C_k \hat{\mathbb{E}}|\int_0^t (B_r^{(i)})^2 d\langle B^{(j)} \rangle_r|^k \leq C_{k,\bar{\sigma}} \hat{\mathbb{E}}|\int_0^t (B_r^{(i)})^2 dr|^k \\ &\leq C_{k,\bar{\sigma}} t^k \hat{\mathbb{E}}(\frac{1}{t} \int_0^t |B_r^{(i)}|^2 dr) \\ &\leq C_{k,\bar{\sigma}} t^{2k}, \end{aligned}$$

where $C_{k,\bar{\sigma}}$ is a constant depending on k and $\bar{\sigma}$, which implies the result by basic inequalities. \square

Since (B, \mathbb{B}) are rough paths $\hat{c} - q.s.$, for $(Y, Y') \in \mathcal{D}_{B(\omega)}^{2\alpha} \cap M_G^2$, we denote $\int Y_r d\mathbf{B}_r$ as the rough integral and $\int Y_r dB_r$ as the Itô integral with respect to G -Brownian motion.

Proposition 3.3. (*G -Itô stochastic integral as rough integral*) Assume $(Y, Y')(\omega) \in \mathcal{D}_{B(\omega)}^{2\alpha}([0, T], \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n))$, $\hat{c} - q.s.$, and $Y, Y' \in M_G^2(0, T)$, with Y_t, Y'_t in $L_G^2(\Omega_t)$, for any $t \in [0, T]$. Furthermore, suppose $\|Y\|_\alpha, \|Y'\|_\alpha \in \mathbb{L}^2 < \infty$. Then the identity holds,

$$\int_0^T Y_r d\mathbf{B}_r = \int_0^T Y_r dB_r, \quad \hat{c} - q.s.. \quad (3.3)$$

In particular, $\sum_{(u,v) \in \mathcal{P}} (Y_u B_{u,v} + Y'_u \mathbb{B}_{u,v})$ converges to $\int_0^T Y_r d\mathbf{B}_r$ in the L_G^2 -norm sense.

Proof. Suppose \mathcal{P} is any partition of $[0, T]$ and set $Y_t^\mathcal{P} := \sum_{[u,v] \in \mathcal{P}} Y_u 1_{[u,v)}(t)$. Then we have inequalities,

$$\begin{aligned} \hat{\mathbb{E}}|\int_0^T (Y_t - Y_t^\mathcal{P}) dB_t|^2 &\leq C_{\bar{\sigma}} \hat{\mathbb{E}} \int_0^T |Y_t - Y_t^\mathcal{P}|^2 dt \leq C_{\bar{\sigma}} \sum_{\mathcal{P}} \int_u^v (t-u)^{2\alpha} \hat{\mathbb{E}}\|Y\|_\alpha^2 dt \\ &\leq C_{\bar{\sigma}} T |\mathcal{P}|^{2\alpha} \hat{\mathbb{E}}\|Y\|_\alpha^2. \end{aligned}$$

In particular, $\sum_{\mathcal{P}} Y_u B_{u,v} \xrightarrow{|\mathcal{P}| \rightarrow 0} \int_0^T Y_t dB_t$, in the sense of L_G^2 -norm, so according to Proposition 2.17, there exists a subsequence, denoted as $\sum_{\mathcal{P}_n} Y_u B_{u,v}$, converging to $\int_0^T Y_r dB_r$, $\hat{c} - q.s.$.

By the definition of rough integral, $\sum_{\mathcal{P}_n} (Y_u B_{u,v} + Y'_u \mathbb{B}_{u,v}) \rightarrow \int_0^T Y_r d\mathbf{B}_r$, $\hat{c} - q.s.$. We claim that, as the difference term of the two sequences, $\sum_{\mathcal{P}_n} Y'_u \mathbb{B}_{u,v}$ converges to 0 in L_G^2 -norm sense, and then according to this, there exists a subsequence $\{\mathcal{P}_{n_k}\}$ such that $\sum_{\mathcal{P}_{n_k}} Y'_u \mathbb{B}_{u,v}$ converges to 0, $\hat{c} - q.s.$, which implies the desired result. At last, one has the following inequalities,

$$\begin{aligned} \|\sum_{\mathcal{P}_{n_k}} Y'_u \mathbb{B}_{u,v}\|_{L_G^2}^2 &= \hat{\mathbb{E}}[\sum_{\mathcal{P}_{n_k}} |Y'_u|^2 |\mathbb{B}_{u,v}|^2] \\ &= \hat{\mathbb{E}}[|Y'_{u_1}|^2 |\mathbb{B}_{u_1, v_1}|^2 + \cdots + |Y'_{u_{l_k}}|^2 \hat{\mathbb{E}}[|\mathbb{B}_{u_{l_k}, v_{l_k}}|^2 | \Omega_{u_{l_k}}]] \end{aligned}$$

$$\begin{aligned}
&\leq \hat{\mathbb{E}}[|Y'_{u_1}|^2 |\mathbb{B}_{u_1, v_1}|^2 + \cdots + |Y'_{u_{l_k-1}}|^2 |\mathbb{B}_{u_{l_k-1}, v_{l_k-1}}|^2 + C|Y'_{u_{l_k}}|^2 (v_{l_k} - u_{l_k})^2] \\
&= \hat{\mathbb{E}}[|Y'_{u_1}|^2 |\mathbb{B}_{u_1, v_1}|^2 + \cdots + \hat{\mathbb{E}}[|Y'_{u_{l_k-1}}|^2 |\mathbb{B}_{u_{l_k-1}, v_{l_k-1}}|^2 + C|Y'_{u_{l_k}}|^2 (v_{l_k} - u_{l_k})^2 | \Omega_{u_{l_k-1}}]] \\
&\leq \hat{\mathbb{E}}[|Y'_{u_1}|^2 |\mathbb{B}_{u_1, v_1}|^2 + \cdots + \hat{\mathbb{E}}[|Y'_{u_{l_k-1}}|^2 |\mathbb{B}_{u_{l_k-1}, v_{l_k-1}}|^2 | \Omega_{u_{l_k-1}}]] + C \hat{\mathbb{E}}[|Y'_{u_{l_k}}|^2 (v_{l_k} - u_{l_k})^2 | \Omega_{u_{l_k-1}}]] \\
&\leq \hat{\mathbb{E}}[|Y'_{u_1}|^2 |\mathbb{B}_{u_1, v_1}|^2 + \cdots + |Y'_{u_{l_k-2}}|^2 |\mathbb{B}_{u_{l_k-2}, v_{l_k-2}}|^2 + C(|Y'_{u_{l_k-1}}|^2 (v_{l_k-1} - u_{l_k-1})^2 + |Y'_{u_{l_k}}|^2 (v_{l_k} - u_{l_k})^2)] \\
&\leq \dots \\
&\leq C \hat{\mathbb{E}}[\sum_{\mathcal{P}_{n_k}} |Y'_u|^2 (v - u)^2] \\
&\leq C |\mathcal{P}_{n_k}| \|Y'\|_{M_G^2}^2.
\end{aligned}$$

The last inequality follows from the convergence, $\sum_{(u,v) \in \mathcal{P}} Y'_u 1_{[u,v)}(t) \xrightarrow{|\mathcal{P}|} Y'$ in the sense of M_G^2 . Indeed,

$$\begin{aligned}
\hat{\mathbb{E}} \int_0^T \left| \sum_{(u,v) \in \mathcal{P}} Y'_u 1_{[u,v)}(t) - Y'_t \right|^2 dt &\leq \sum_{\mathcal{P}} \hat{\mathbb{E}} \int_u^v |Y'_u - Y'_t|^2 dt \\
&\leq \hat{\mathbb{E}} \|Y'\|_{\alpha}^2 T |\mathcal{P}|^{2\alpha} / (2\alpha + 1) \rightarrow 0.
\end{aligned}$$

□

Remark 3.4. Here $Y, Y' \in M_G^2$ means every element of Y and Y' belongs to M_G^2 . According to the above proof, one can simply check that the assumption $\|Y'\|_{\alpha} \|Y'\|_{\mathbb{L}^2} < \infty$ could be replaced by $|Y'|$ bounded.

Example 3.1. (i). For fixed $\alpha \in (\frac{1}{3}, \frac{1}{2})$ and any function $F \in \mathcal{C}^2(\mathbb{R}, \mathbb{R})$ with polynomial growth for the first and second order derivatives, i.e.,

$$|D^2 F(x)| + |DF(x)| \leq C(1 + |x|^k),$$

for some positive constants C, k , it is simple to check that $(Y, Y') := (F(B), DF(B))$, where B is a one-dimensional G -Brownian motion, satisfies the assumption in the above proposition. Indeed, according to Taylor's expansion,

$$\begin{aligned}
F(B_t) - F(B_s) &= DF(B_s + \lambda_1(B_t - B_s))(B_t - B_s), \\
DF(B_t) - DF(B_s) &= D^2 F(B_s + \lambda_2(B_t - B_s))(B_t - B_s), \\
F(B_t) - F(B_s) &= DF(B_s)(B_t - B_s) + \frac{1}{2} D^2 F(B_s + \lambda_3(B_t - B_s))(B_t - B_s)^2,
\end{aligned}$$

for some $\lambda_i(\omega) \in [0, 1], i = 1, 2, 3$.

By Theorem 3.1, it holds that,

$$\begin{aligned}
\|F(B)\|_{\alpha} &\leq \sup_{\substack{s, t \in [0, T] \\ \lambda_1 \in [0, 1]}} |DF(B_s + \lambda_1(B_t - B_s))| \|B\|_{\alpha}, \quad \hat{c} - q.s.; \\
\|DF(B)\|_{\alpha} &\leq \sup_{\substack{s, t \in [0, T] \\ \lambda_2 \in [0, 1]}} |DF(B_s + \lambda_2(B_t - B_s))| \|B\|_{\alpha}, \quad \hat{c} - q.s.; \\
\|R^Y\|_{2\alpha} &\leq \frac{1}{2} \sup_{\substack{s, t \in [0, T] \\ \lambda_3 \in [0, 1]}} |D^2 F(B_s + \lambda_3(B_t - B_s))| \|B\|_{\alpha}^2, \quad \hat{c} - q.s.,
\end{aligned}$$

so $(F(B), DF(B)) \in \mathcal{D}_{B(\omega)}^{2\alpha}([0, T], \mathbb{R}), \hat{c} - q.s.$. Furthermore, by the polynomial growth condition and Theorem 3.1, one can simply check that

$$\|Y\|_{\alpha} \|Y'\|_{\alpha} \|Y'\|_{\mathbb{L}^2} < \infty.$$

(ii). For a given function $f \in C^1(\mathbb{R}, \mathbb{R})$, which satisfies

$$|f(x)| + |Df(x)| \leq K(1 + |x|^d),$$

for some positive constants K, d , define $(Z, Z') := (\int_0^\cdot f(B_r)dB_r, f(B_\cdot))$. Firstly, we need to show $Z \in M_G^2(0, T)$. Define $Z_t^N := \sum_{i=0}^{k_N} Z_{t_i} 1_{[t_i^N, t_{i+1}^N)}(t)$, where $\mathcal{P}^N := \{0 = t_0^N < t_1^N < \dots < t_{k_N}^N = T\}$ is any sequence of partition with modulus $|\mathcal{P}^N|$ converging to 0, and then one could obtain $Z^N \xrightarrow{N} Z$ under the norm of M_G^2 . Indeed,

$$\begin{aligned} \hat{\mathbb{E}} \int_0^T (Z_t^N - Z_t)^2 dt &\leq \sum_{\mathcal{P}^N} \int_{t_i^N}^{t_{i+1}^N} \hat{\mathbb{E}} (Z_t^N - Z_t)^2 dt \\ &= \sum_{\mathcal{P}^N} \int_{t_i^N}^{t_{i+1}^N} \hat{\mathbb{E}} \left(\int_{t_i^N}^t f(B_r) dB_r \right)^2 dt \\ &\leq C_{\bar{\sigma}, K, d, T} |\mathcal{P}^N| \rightarrow 0, \end{aligned}$$

where $C_{\bar{\sigma}, K, d}$ is a constant depending on $\bar{\sigma}, K, d$. Secondly, one needs to check that $(Z, Z') \in \mathcal{D}_{B(\omega)}^{2\alpha}([0, T], \mathbb{R}), \hat{c} - q.s.$. According to Theorem 3.1, it is simple to obtain that $Z \in \mathcal{C}^\alpha, \hat{c} - q.s.$, and $\|\| \|Z\|_\alpha\|_{\mathbb{L}^q} < \infty$, for any $q \geq 2$ and $\alpha \in (\frac{1}{3}, \frac{1}{2} - \frac{1}{q})$. Finally, only $R^Z \in \mathcal{C}^{2\alpha}$ needs to be checked. Define $H(x) := \int_0^x f(y)dy$. Then $DH(x) = f(x)$, and $H(x)$ has polynomial growth for the first and second derivatives. By G -Itô's formula,

$$H(B_t) - H(B_s) = Z_{s,t} + \frac{1}{2} \int_s^t Df(B_r) d\langle B \rangle_r, \quad \hat{c} - q.s..$$

According to example (i), $R_{s,t}^{H(B)} := H(B_t) - H(B_s) - f(B_s)B_{s,t}$ quasi-surely belongs to $\mathcal{C}^{2\alpha}$. Since $\langle B \rangle$ is absolutely continuous, one could say $R_{s,t}^Z := Z_{s,t} - f(B_s)B_{s,t} \in \mathcal{C}^{2\alpha}, \hat{c} - q.s..$

Remark 3.5. It is easy to see that one could replace B with Itô processes and apply similar tricks for more examples.

3.2 G -Stratonovich integral as rough integral

Firstly, we provide a description of Stratonovich integral with respect to G -Brownian motion. Define $\langle Y, B^{(k)} \rangle_t := \lim_{|\mathcal{P}| \rightarrow 0} \sum_{(u,v) \in \mathcal{P}} Y_{u,v} B_{u,v}^{(k)}$, for any $k = 1, \dots, d$, whenever the limit exists in $L_G^1(\Omega_t)$, for any $t \in [0, T]$.

Proposition 3.6. For any $\beta = (\beta^{(1)}, \dots, \beta^{(d)}) \in M_G^2$, define $Y_t := \int_0^t \beta_r^{(l)} dB_r^{(l)}$. Then one has

$$\langle Y, B^{(k)} \rangle_t = \int_0^t \beta_r^{(l)} d\langle B^{(l)}, B^{(k)} \rangle_r, \quad \hat{c} - q.s.. \quad (3.4)$$

Proof. By linearity one only needs to show the case that β is one-dimensional, i.e. $Y_t = \int_0^t \beta_r dB_r^{(l)}$, for any fixed $l = 1, \dots, d$.

Step1: Suppose that $\beta_s \in M_G^{2,0}$, with the form $\beta_s = \sum_{i=0}^{N-1} \xi_i 1_{[t_i, t_{i+1})}(s)$, $|\xi_i| \leq K, i = 0 \dots N-1$, and the partition $\mathcal{Q} := \{0 = t_0 < t_1 < t_2 < \dots < t_N = t\}$ fixed.

For any partition $\mathcal{P} = \{0 = \tau_0 < \tau_1 < \tau_2 < \dots < \tau_M = t\}$, satisfying $|\mathcal{P}| \leq |\mathcal{Q}|$, it holds that

$$\begin{aligned} \sum_{\mathcal{P}} (Y_{\tau_i, \tau_{i+1}} B_{\tau_i, \tau_{i+1}}^{(k)}) &= \sum_{\mathcal{P}} \left(\int_{\tau_i}^{\tau_{i+1}} \beta_s dB_s^{(l)} B_{\tau_i, \tau_{i+1}}^{(k)} \right) \\ &= \sum_{[\tau_i, \tau_{i+1}) \subset [t_j, t_{j+1})} (\xi_j B_{\tau_i, \tau_{i+1}}^{(l)}) B_{\tau_i, \tau_{i+1}}^{(k)} \\ &+ \sum_{t_j \in [\tau_i, \tau_{i+1})} (\xi_{j-1} (B_{t_j}^{(l)} - B_{\tau_i}^{(l)}) + \xi_j (B_{\tau_{i+1}}^{(l)} - B_{t_j}^{(l)})) B_{\tau_i, \tau_{i+1}}^{(k)} \end{aligned}$$

$$\begin{aligned}
&= \sum_{[\tau_i, \tau_{i+1}) \subset [t_j, t_{j+1})} (\xi_j B_{\tau_i, \tau_{i+1}}^{(l)}) B_{\tau_i, \tau_{i+1}}^{(k)} \\
&+ \sum_{t_j \in [\tau_i, \tau_{i+1})} (\xi_{j-1} (B_{t_j}^{(l)} - B_{\tau_i}^{(l)}) + \xi_j (B_{\tau_{i+1}}^{(l)} - B_{t_j}^{(l)})) (B_{\tau_i, t_j}^{(k)} + B_{t_j, \tau_{i+1}}^{(k)}) \\
&= \sum_{\mathcal{P} \vee \mathcal{Q}} (\xi_u B_{u,v}^{(l)}) B_{u,v}^{(k)} + \sum_{j=1}^{N-1} \xi_{j-1} B_{\tau_i, t_j}^{(l)} B_{t_j, \tau_{i+1}}^{(k)} + \sum_{j=1}^{N-1} \xi_j B_{\tau_i, t_j}^{(k)} B_{t_j, \tau_{i+1}}^{(l)}, \quad \hat{c} - q.s.,
\end{aligned}$$

in the last equation of which we patch the two partitions together.

According to Chapter 3 Lemma 4.6 in [24], it suffices to show the convergence

$$\sum_{j=1}^{N-1} \xi_{j-1} B_{\tau_i, t_j}^{(l)} B_{t_j, \tau_{i+1}}^{(k)} + \sum_{j=1}^{N-1} \xi_j B_{\tau_i, t_j}^{(k)} B_{t_j, \tau_{i+1}}^{(l)} \xrightarrow{|\mathcal{P}|} 0,$$

in the sense of L_G^1 . Indeed,

$$\hat{\mathbb{E}} \left[\left| \sum_{j=1}^{N-1} \xi_{j-1} B_{\tau_i, t_j}^{(l)} B_{t_j, \tau_{i+1}}^{(k)} + \sum_{j=1}^{N-1} \xi_j B_{\tau_i, t_j}^{(k)} B_{t_j, \tau_{i+1}}^{(l)} \right| \right] \leq 2(N-1)K|\mathcal{P}| \rightarrow 0.$$

Step2: For any $\beta \in M_G^2$, assume $\{\beta_s^n\}_n \subset M_G^{2,0}$, converges to β in the sense of M_G^2 . One has inequalities,

$$\begin{aligned}
&\hat{\mathbb{E}} \left[\left| \sum_{\mathcal{P}} \int_u^v \beta_s dB_s^{(l)} B_{u,v}^{(k)} - \int_0^t \beta_s d\langle B^{(l)}, B^{(k)} \rangle_s \right| \right] \\
&\leq \hat{\mathbb{E}} \left[\left| \sum_{\mathcal{P}} \left(\int_u^v \beta_s dB_s^{(l)} B_{u,v}^{(k)} - \int_u^v \beta_s^n dB_s^{(l)} B_{u,v}^{(k)} + \int_u^v \beta_s^n dB_s^{(l)} B_{u,v}^{(k)} \right. \right. \right. \\
&\quad \left. \left. - \int_u^v \beta_s^n d\langle B^{(l)}, B^{(k)} \rangle_s + \int_u^v \beta_s^n d\langle B^{(l)}, B^{(k)} \rangle_s - \int_u^v \beta_s d\langle B^{(l)}, B^{(k)} \rangle_s \right) \right| \right] \\
&\leq \hat{\mathbb{E}} \left[\left| \sum_{\mathcal{P}} \int_u^v \beta_s dB_s^{(l)} B_{u,v}^{(k)} - \int_u^v \beta_s^n dB_s^{(l)} B_{u,v}^{(k)} \right| \right] + \hat{\mathbb{E}} \left[\left| \sum_{\mathcal{P}} \int_u^v \beta_s^n dB_s^{(l)} B_{u,v}^{(k)} \right. \right. \\
&\quad \left. \left. - \int_u^v \beta_s^n d\langle B^{(l)}, B^{(k)} \rangle_s \right| \right] + \hat{\mathbb{E}} \left[\left| \sum_{\mathcal{P}} \int_u^v (\beta_s^n - \beta_s) d\langle B^{(l)}, B^{(k)} \rangle_s \right| \right] \tag{3.5}
\end{aligned}$$

The second term in (3.5) converges to 0 by Step1. According to definitions, the third term also converges to 0.

At last, for the first term, since the calculation is carried in L_G^2 and B is a martingale under each $P \in \mathcal{P}$, one obtains that

$$\begin{aligned}
&\hat{\mathbb{E}} \left[\left| \sum_{\mathcal{P}} \int_u^v (\beta_s - \beta_s^n) dB_s^{(l)} B_{u,v}^{(k)} \right| \right] \\
&\leq \sup_{P \in \mathcal{P}} \sum_{\mathcal{P}} E_P \left[\left| \int_u^v (\beta_s - \beta_s^n) dB_s^{(l)} B_{u,v}^{(k)} \right| \right] \\
&\leq \sup_{P \in \mathcal{P}} \bar{\sigma}^2 \sum_{\mathcal{P}} \{ E_P [\int_u^v (\beta_s - \beta_s^n)^2 ds] \}^{\frac{1}{2}} |v - u|^{\frac{1}{2}} \\
&\leq \bar{\sigma}^2 \sup_{P \in \mathcal{P}} T^{\frac{1}{2}} \{ \sum_{\mathcal{P}} E_P [\int_u^v (\beta_s - \beta_s^n)^2 ds] \}^{\frac{1}{2}} \\
&\leq \bar{\sigma}^2 T^{\frac{1}{2}} \{ \hat{\mathbb{E}} [\int_0^T (\beta_s - \beta_s^n)^2 ds] \}^{\frac{1}{2}} \rightarrow 0.
\end{aligned}$$

□

Corollary 3.7. For

$$Y_t = \xi + \int_0^t \beta_s^{(j)} dB_s^{(j)} + \int_0^t \alpha_s ds + \int_0^t \gamma_s^{(j,l)} d\langle B^{(j)}, B^{(l)} \rangle_s, \quad (3.6)$$

with $\beta \in M_G^2$, and $\alpha, \gamma \in M_G^{1+\delta}(0, T)$, for some $\delta > 0$, one has the expression,

$$\langle Y, B^{(k)} \rangle_t = \int_0^t \beta_s^{(j)} d\langle B^{(j)}, B^{(k)} \rangle_s, \quad \hat{c} - q.s..$$

Proof. Without loss of generality, the proof can be done by showing

$$\sum_{(u,v) \in \mathcal{P}} \int_u^v \gamma_s d\langle B^{(j)} \rangle_s B_{u,v}^{(k)} \xrightarrow{|\mathcal{P}|} 0, \quad \sum_{(u,v) \in \mathcal{P}} \int_u^v \alpha_s ds B_{u,v}^{(k)} \xrightarrow{|\mathcal{P}|} 0,$$

in the sense of L_G^1 . We only show the first convergence. Indeed, by boundedness for $\frac{d\langle B^{(j)} \rangle_t}{dt}$, one has the following inequalities

$$\begin{aligned} & \hat{\mathbb{E}} \left[\left| \sum_{\mathcal{P}} \int_u^v \gamma_s d\langle B^{(j)} \rangle_s B_{u,v}^{(k)} \right| \right] \leq \bar{\sigma}^2 \sup_{P \in \mathcal{P}} \sum_P E_P \left[|B_{u,v}^{(k)}| \left| \int_u^v \gamma_s ds \right| \right] \\ & \leq \bar{\sigma}^2 \sup_{P \in \mathcal{P}} \sum_P [E_P \left| \int_u^v \gamma_s ds \right|^{1+\delta}]^{\frac{1}{1+\delta}} [E_P |B_{u,v}^{(k)}|^{\frac{1+\delta}{\delta}}]^{\frac{\delta}{1+\delta}} \\ & \leq \bar{\sigma}^3 \sup_{P \in \mathcal{P}} \sum_P [E_P \int_u^v |\gamma_s|^{1+\delta} ds]^{\frac{1}{1+\delta}} |v - u|^{\frac{\delta}{1+\delta}} |v - u|^{\frac{1}{2}} \\ & \leq \bar{\sigma}^3 \sup_{P \in \mathcal{P}} |\mathcal{P}|^{\frac{1}{2}} \left[\sum_P E_P \int_u^v |\gamma_s|^{1+\delta} ds \right]^{\frac{1}{1+\delta}} T^{\frac{\delta}{1+\delta}} \\ & \leq \bar{\sigma}^3 T^{\frac{\delta}{1+\delta}} \|\gamma\|_{M_G^{1+\delta}} |\mathcal{P}|^{\frac{1}{2}} \rightarrow 0. \end{aligned}$$

□

Definition 3.8. (*Stratonovich integration with respect to G-Brownian motion*) Suppose $Y = (Y_1, \dots, Y_d) \in M_G^2(0, T)$, and $\langle Y^{(i)}, B^{(i)} \rangle$ exist for any i . The Stratonovich integral of Y against B , with value in L_G^1 , is given by identity:

$$\int_0^t Y_s^{(i)} \circ dB_s^{(i)} := \int_0^t Y_s^{(i)} dB_s^{(i)} + \frac{1}{2} \langle Y^{(i)}, B^{(i)} \rangle_t, \quad \hat{c} - q.s.. \quad (3.7)$$

Proposition 3.9. Assume Y defined as $Y_t := \int_0^t \beta_s^{(j)} dB_s^{(j)}$, with $\beta \in M_G^2$. Then, for partitions \mathcal{P} of $[0, t]$ with $|\mathcal{P}| \rightarrow 0$, it holds that

$$L_G^1 - \lim_{|\mathcal{P}| \rightarrow 0} \sum_{(u,v) \in \mathcal{P}} \frac{Y_u + Y_v}{2} B_{u,v}^{(k)} = \int_0^t Y_s \circ dB_s^{(k)} \quad (3.8)$$

Proof. Suppose $t = T$ here. According to the definition of $\langle Y, B^{(k)} \rangle$, it suffices to show the following convergence under the case that β is one-dimensional,

$$\sum_{\mathcal{P}} Y_u B_{u,v}^{(k)} \xrightarrow{L_G^1} \int_0^T Y_r dB_r^{(k)}.$$

Step1. If $\beta_s = \sum_{i=0}^{N-1} \xi_i 1_{[t_i, t_{i+1})}(s)$, with $\mathcal{Q} := 0 = t_0 < t_1 < \dots < t_N = T$, a fixed partition, one has the identity

$$Y_r = \int_0^r \beta_s dB_s^{(l)}$$

$$\begin{aligned}
&= \sum_{i=0}^{N-2} \xi_i B_{t_i, t_{i+1}}^{(l)} 1_{[t_{i+1}, T)}(r) + \sum_{i=0}^{N-1} \xi_i B_{t_i, r}^{(l)} 1_{[t_i, t_{i+1})}(r) + \xi_{N-1} B_{t_{N-1}, t_N}^{(l)} 1_{\{T\}}(r) \\
&= \sum_{i=0}^{N-1} \left(\sum_{j=0}^{i-1} \xi_j B_{t_j, t_{j+1}}^{(l)} - \xi_i B_{t_i}^{(l)} \right) 1_{[t_i, t_{i+1})}(r) + B_r^{(l)} \sum_{i=0}^{N-1} \xi_i 1_{[t_i, t_{i+1})}(r) + \xi_{N-1} B_{t_{N-1}, t_N}^{(l)} 1_{\{T\}}(r) \\
&= \sum_{i=0}^{N-1} \tilde{\xi}_i 1_{[t_i, t_{i+1})}(r) + B_r^{(l)} \sum_{i=0}^{N-1} \xi_i 1_{[t_i, t_{i+1})}(r) + \xi_{N-1} B_{t_{N-1}, t_N}^{(l)} 1_{\{T\}}(r),
\end{aligned}$$

where we denote $\tilde{\xi}_i := (\sum_{j=0}^{i-1} \xi_j B_{t_j, t_{j+1}}^{(l)} - \xi_i B_{t_i}^{(l)})$, and $\sum_{j=0}^{-1} \xi_j B_{t_j, t_{j+1}}^{(l)} = 0$.

It follows that

$$\begin{aligned}
\int_0^T Y_r dB_r^{(k)} &= \sum_{i=0}^{N-1} \left(\sum_{j=0}^{i-1} \xi_j B_{t_j, t_{j+1}}^{(l)} - \xi_i B_{t_i}^{(l)} \right) B_{t_i, t_{i+1}}^{(k)} + \sum_{i=0}^{N-1} \xi_i \int_{t_i}^{t_{i+1}} B_r^{(l)} dB_r^{(k)} \\
&= \sum_{i=0}^{N-1} \tilde{\xi}_i B_{t_i, t_{i+1}}^{(k)} + \sum_{i=0}^{N-1} \xi_i \int_{t_i}^{t_{i+1}} B_r^{(l)} dB_r^{(k)}, \quad \hat{c} - q.s.,
\end{aligned} \tag{3.9}$$

On the other hand, suppose $\mathcal{P} := \{0 = \tau_0 < \tau_1 < \dots < \tau_M = T\}$. It holds that

$$\begin{aligned}
&\sum_{j=0}^{M-1} Y_{\tau_j} B_{\tau_j, \tau_{j+1}}^{(k)} \\
&= \sum_{j=0}^{M-1} \left(\sum_{i=0}^{N-1} \tilde{\xi}_i 1_{[t_i, t_{i+1})}(\tau_j) B_{\tau_j, \tau_{j+1}}^{(k)} \right) \\
&+ \sum_{j=0}^{M-1} B_{\tau_j}^{(l)} \left(\sum_{i=0}^{N-1} \xi_i 1_{[t_i, t_{i+1})}(\tau_j) \right) B_{\tau_j, \tau_{j+1}}^{(k)}, \quad \hat{c} - q.s.
\end{aligned} \tag{3.10}$$

We claim that the first part of (3.10) converges to the first part of (3.9) in L_G^1 -norm sense, and the second part of (3.10) also does converge to the last part of (3.9).

Firstly, for any $i = 0, \dots, N-1$, assume τ_{k_i} is the first endpoint in partition \mathcal{P} entering the interval $[t_i, t_{i+1})$. Note that $k_i \geq 1$, once making sure $|\mathcal{P}| < |\mathcal{Q}|$. Then it turns out that

$$\begin{aligned}
&\sum_{j=0}^{M-1} \left(\sum_{i=0}^{N-1} \tilde{\xi}_i 1_{[t_i, t_{i+1})}(\tau_j) B_{\tau_j, \tau_{j+1}}^{(k)} \right) \\
&= \sum_{i=0}^{N-1} \tilde{\xi}_i B_{t_i, t_{i+1}}^{(k)} + \sum_{i=0}^{N-1} (\tilde{\xi}_i B_{\tau_{k_i}, t_i}^{(k)} + \tilde{\xi}_i B_{\tau_{k_i+1}-1, t_{i+1}}^{(k)}).
\end{aligned} \tag{3.11}$$

A similar argument as Lemma 3.6 shows that the second part of (3.11) converges to 0 in the L_G^1 -norm sense.

The convergence of the second part of (3.10) follows from the fact that

$$L_G^2 - \sum_{\mathcal{P} \cap [t_i, t_{i+1})} B_u^{(l)} B_{u,v}^{(k)} \xrightarrow{|\mathcal{P}|} \int_{t_i}^{t_{i+1}} B_r^{(l)} dB_r^{(k)}.$$

Step2. According to the definition of M_G^2 , for any $Y_t := \int_0^t \beta_s dB_s^{(l)}$, with $\beta \in M_G^2$, there exists $\{\beta^n\}_{n=1} \in M_G^{2,0}$, such that $\beta^n \xrightarrow{M_G^2} \beta$. Then one has the following identity by inserting terms

$$\sum_{\mathcal{P}} Y_u B_{u,v}^{(k)} - \int_0^T Y_t dB_t^{(k)}$$

$$= \sum_{\mathcal{P}} (Y_u B_{u,v}^{(k)} - \int_0^u \beta_s^n dB_s^{(l)}) B_{u,v}^{(k)} \quad (3.12)$$

$$+ \sum_{\mathcal{P}} \int_0^u \beta_s^n dB_s^{(l)} B_{u,v}^{(k)} - \int_0^T Y_t^n dB_t^{(k)} \quad (3.13)$$

$$+ \int_0^T Y_t^n dB_t^{(k)} - \int_0^T Y_t dB_t^{(k)}, \quad \hat{c} - q.s., \quad (3.14)$$

where we denote $Y_t^n := \int_0^t \beta_s^n dB_s^{(l)}$.

We claim that (3.12), (3.13), (3.14) converge to 0 in the sense of L_G^1 .

Firstly, for (3.14), it suffices to prove that $Y_t^n \xrightarrow{M_G^2} Y_t$, which follows directly from

$$\hat{\mathbb{E}} \int_0^T \left| \int_0^t (\beta_s^n - \beta_s) dB_s^{(l)} \right|^2 dt \leq \bar{\sigma}^2 T \hat{\mathbb{E}} \int_0^T |\beta_s^n - \beta_s|^2 ds \rightarrow 0$$

as n goes to infinity.

Secondly, for a fixed n , according to Step1, (3.13) converges to 0 as $|\mathcal{P}| \rightarrow 0$.

Thirdly, for (3.12), it holds that

$$\begin{aligned} \hat{\mathbb{E}} \left| \sum_{\mathcal{P}} (Y_u - \int_0^u \beta_s^n dB_s^{(l)}) B_{u,v}^{(k)} \right|^2 &= \hat{\mathbb{E}} \sum_{\mathcal{P}} |Y_u - \int_0^u \beta_s^n dB_s^{(l)}|^2 |B_{u,v}^{(k)}|^2 \\ &\leq \bar{\sigma}^2 \sum_{\mathcal{P}} |v - u| \hat{\mathbb{E}} \left| \int_0^u (\beta_s^n - \beta_s)^2 d\langle B^{(l)} \rangle_s \right| \\ &\leq \bar{\sigma}^4 T \hat{\mathbb{E}} \int_0^T |\beta_s^n - \beta_s|^2 ds \rightarrow 0 \end{aligned}$$

□

Corollary 3.10. Suppose Y_t defined as (3.6), with $\beta, \alpha, \gamma \in M_G^2$. Then it holds that

$$L_G^1 - \lim_{|\mathcal{P}| \rightarrow 0} \sum_{(u,v) \in \mathcal{P}} \frac{Y_u + Y_v}{2} B_{u,v}^{(k)} = \int_0^T Y_s \circ dB_s^{(k)}.$$

Proof. By the above proposition and linearity of integration, it suffices to show the convergence of $\sum_{\mathcal{P}} Y_u B_{u,v}^{(k)}$ to $\int_0^T Y_t dB_t^{(k)}$, in the case that $Y_t = \int_0^t \alpha_s ds$. Indeed, with an application of Fubini's theorem, one has inequalities

$$\begin{aligned} & \left| \hat{\mathbb{E}} \left| \int_0^T Y_t dB_t^{(k)} - \sum_{\mathcal{P}} Y_u B_{u,v}^{(k)} \right| \right| \leq \hat{\mathbb{E}} \left| \sum_{\mathcal{P}} \int_u^v \left(\int_u^t \alpha_s ds \right) dB_t^{(k)} \right| \\ & \leq \sup_{P \in \mathcal{P}} \bar{\sigma} \sum_{\mathcal{P}} [E_P \left| \int_u^v \left(\int_u^t \alpha_s ds \right)^2 dt \right|]^{\frac{1}{2}} \\ & \leq \sup_{P \in \mathcal{P}} \bar{\sigma} \sum_{\mathcal{P}} [E_P \int_u^v \left(\int_u^t \alpha_s^2 ds \right) (t - u) dt]^{\frac{1}{2}} \\ & \leq \bar{\sigma} \sup_{P \in \mathcal{P}} T^{\frac{1}{2}} \left[\sum_{\mathcal{P}} E_P \int_u^v \left(\int_u^t \alpha_s^2 ds \right) \frac{t - u}{v - u} dt \right]^{\frac{1}{2}} \\ & \leq \bar{\sigma} T^{\frac{1}{2}} \sup_{P \in \mathcal{P}} \left[\sum_{\mathcal{P}} E_P \int_u^v \alpha_s^2 \left(\frac{v - u}{2} - \frac{(s - u)^2}{2(v - u)} \right) ds \right]^{\frac{1}{2}} \\ & \leq \bar{\sigma} T^{\frac{1}{2}} |\mathcal{P}|^{\frac{1}{2}} \|\alpha\|_{M_G^2}, \end{aligned}$$

which implies the expected conclusion.

□

Remark 3.11. Of course one can further consider the quadratic variation of two G -Itô processes, and obtain similar results. However, by now, we already have got enough information to consider Stratonovich integrals as rough integrals.

In the case where $Y_s = B_s$, one may define the Stratonovich integral with respect to G -Brownian motion,

$$\mathbb{B}_{s,t}^{strat} := \int_s^t B_{s,u} \circ dB_u = \mathbb{B}_{s,t} + \frac{1}{2} \langle B \rangle_{s,t}.$$

Here $\langle B \rangle = \{\langle B^{(i)}, B^{(j)} \rangle\}_{i,j}$ is the variation matrix. According to Theorem 3.1, $\mathbf{B}^{strat} := (B, \mathbb{B}^{strat})$ is also quasi-surely rough paths.

Corollary 3.12. (*G -Stratonovich integral as rough integral*)

Assume $(Y, Y')(\omega) \in \mathcal{D}_{B(\omega)}^{2\alpha}([0, T], \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n))$, $\hat{c} - q.s.$, and $Y, Y' \in M_G^2(0, T)$, with values Y_t, Y'_t in $L_G^2(\Omega_t)$, for any $t \in [0, T]$. Furthermore, suppose $\|Y\|_\alpha, \|Y'\|_\alpha, \|R^Y\|_{2\alpha} < \infty$. Then one has the identity,

$$\langle Y, B \rangle_t = \int_0^t Y'_s d\langle B \rangle_s, \quad \hat{c} - q.s..$$

Moreover, it holds that

$$\int_0^t Y_s d\mathbf{B}^{strat} = \int_0^t Y_s \circ dB_s, \quad \hat{c} - q.s..$$

In particular, the rough integral $\int_0^t Y_s d\mathbf{B}^{strat}$ belongs to L_G^1 .

Proof. Note that

$$\sum_{(u,v) \in \mathcal{P}} Y_{u,v}(B_{u,v}) = \sum_{(u,v) \in \mathcal{P}} (Y'_u B_{u,v})(B_{u,v}) + \sum_{(u,v) \in \mathcal{P}} R_{u,v}^Y(B_{u,v}).$$

By similar tricks applied in the proof of Lemma 3.6 and integrability of $\|Y'\|_\alpha, \|R^Y\|_{2\alpha}$, one could obtain that

$$\sum_{(u,v) \in \mathcal{P}} (Y'_u B_{u,v})(B_{u,v}) \rightarrow \int_0^t Y'_s d\langle B \rangle_s; \quad \sum_{(u,v) \in \mathcal{P}} R_{u,v}^Y(B_{u,v}) \rightarrow 0$$

in the sense of L_G^2 . Then we got the existence of $\langle Y, B \rangle$, i.e. the following identity,

$$\langle Y, B \rangle_t = \int_0^t Y'_s d\langle B \rangle_s, \quad \hat{c} - q.s..$$

By the definition of \mathbf{B}^{strat} and rough integrals, it holds that

$$\int_0^t Y_s d\mathbf{B}^{strat} = \int_0^t Y_s d\mathbf{B}_s + \int_0^t Y'_s d\langle B \rangle_s, \quad \hat{c} - q.s..$$

Then the conclusion follows. \square

4 Roughness of G -Brownian Motion and the Norris Lemma

To build the Norris lemma in G -framework through rough paths, we need to show the θ -Hölder roughness of G -Brownian motion, i.e. $\hat{c}(L_\theta(B) = 0) = 0$, for any $\theta > \frac{1}{2}$. The main idea for the proof of the result (i.e. Proposition 4.4) is adapted from Proposition 6.11 in Chapter 6 of [3].

Lemma 4.1. (*exponential inequality*) Suppose B_t be a d -dimensional G -Brownian motion. One has the following inequality

$$\hat{c}(\sup_{[0,T]} |B_t| \geq \frac{1}{\varepsilon}) \leq d \exp\left(-\frac{1}{\varepsilon^2 d T \bar{\sigma}^2}\right) \quad (4.1)$$

Proof. By the representation for $\hat{\mathbb{E}}$, it holds that

$$\begin{aligned} \hat{c}(\sup_{[0,T]} |B_t| \geq \frac{1}{\varepsilon}) &= \sup_{\alpha \in \mathcal{A}^\Gamma} P_0(\sup_{[0,T]} |\int_0^t \alpha_s dB_s|^2 \geq \frac{1}{\varepsilon^2}) \\ &\leq \sum_i \sup_{\alpha \in \mathcal{A}^\Gamma} P_0(\sup_{[0,T]} |\sum_j \int_0^t \alpha_s^{(i,j)} dB_s^j|^2 \geq \frac{1}{d\varepsilon^2}) \\ &\leq d \exp(-\frac{1}{\varepsilon^2 d T \bar{\sigma}^2}), \end{aligned}$$

where P_0 is the Wiener measure, and classical Bernstein inequality (see p.153 in [25] for example) is applied in the last inequality. \square

Remark 4.2. About large deviation results in G -framework, we refer readers to [8] for more details.

The θ -Hölder roughness of the classical Brownian motion was proposed and proved in [9], which gives a quantitative version of the true roughness of Brownian motion, i.e.,

$$\overline{\lim}_{t \rightarrow s} \frac{|B_{s,t}|}{|t-s|^\theta} = \infty, \quad a.s.,$$

when $\theta > \frac{1}{2}$ (see [4] for the definition of true roughness).

Lemma 4.3. Let B_t be a d -dimensional G -Brownian motion. Then there exists positive constants b, A , depending only on the dimension d , such that for any $\varepsilon \in (0, 1)$, one has the bound

$$\hat{c}(\inf_{\substack{|a|=1 \\ a \in \mathbb{R}^d}} \sup_{t \in [0,T]} |(a \cdot B_t)| \leq \varepsilon) \leq A(\exp(-bT\underline{\sigma}^2\varepsilon^{-2}) + \exp(-bT^{-1}(\bar{\sigma}\varepsilon)^{-2})). \quad (4.2)$$

Proof. Note that $B_t^a := a \cdot B_t$ is a G_{aa^T} -Brownian motion, with $\bar{\sigma}_{aa^T}^2 = 2G(aa^T) = \hat{\mathbb{E}}[a^T \langle B \rangle_1 a] \geq \underline{\sigma}^2 |a|^2 = \underline{\sigma}^2$. Here $\underline{\sigma}$ is positive as introduced in Part 2. According to small ball estimates for G -Brownian motion, i.e. Lemma 6.1 in [27], one has the bound

$$\sup_{|a|=1} \hat{c}(\sup_{t \in [0,T]} |(a \cdot B_t)| \leq \varepsilon) \leq \frac{4}{\pi} \exp(-\frac{T\pi^2 \underline{\sigma}^2}{8\varepsilon^2}),$$

for any $\varepsilon \in (0, 1)$. Now cover the sphere $|a| = 1$ with at most $D\varepsilon^{-2d}$ balls of radius ε^2 centered at a_i , D a constant depending on how to divide the sphere or the ball. By applying Lemma 4.1, one obtains inequalities

$$\begin{aligned} \hat{c}(\inf_{|a|=1} \sup_{[0,T]} |(a \cdot B_t)| \leq \varepsilon) &\leq \sum_{i=1}^{D\varepsilon^{-2d}} \hat{c}(\inf_{\substack{a \in O(a_i, \varepsilon^2) \\ |a|=1}} \sup_{t \in [0,T]} |(a \cdot B_t)| \leq \varepsilon) \\ &\leq D\varepsilon^{-2d} [\sup_{|a|=1} \hat{c}(\sup_{[0,T]} |(a \cdot B_t)| \leq 2\varepsilon) + \hat{c}(\sup_{[0,T]} |B_t| \geq \frac{1}{\varepsilon})] \\ &\leq A(\exp(-bT\underline{\sigma}^2\varepsilon^{-2}) + \exp(-bT^{-1}\bar{\sigma}^{-2}\varepsilon^{-2})). \end{aligned}$$

\square

Proposition 4.4. (Hölder roughness for G -Brownian motion) Let B be a d -dimensional G -Brownian motion. Then for any $\theta \in (\frac{1}{2}, 1)$, $B(\omega)$ is θ -Hölder rough, \hat{c} - $q.s.$ with scale $\frac{T}{2}$. More precisely, there exist positive constants K, l , depending on $T, \bar{\sigma}, \underline{\sigma}$, such that for any $\tilde{\varepsilon} \in (0, \frac{1}{2T^\theta})$, one has the bound

$$\hat{c}(L_\theta(B) < \tilde{\varepsilon}) \leq K \exp(-l\tilde{\varepsilon}^{-2}). \quad (4.3)$$

Proof. Define $D_\theta(B) := \inf_{|a|=1, n \geq 1, k \leq 2^n} \sup_{s, t \in [\frac{k-1}{2^n}T, \frac{k}{2^n}T]} 2^{\theta n} |(a \cdot B_{s,t})|$. Then for any fixed a, s, ε , with $|a| = 1, s \in [0, T]$, and $\varepsilon \in (0, \frac{T}{2})$, there exist $n, k \in \mathbb{N}$, such that $\frac{T}{2^n} < \varepsilon \leq \frac{T}{2^{n-1}}$, and $I_{k,n} := [\frac{k-1}{2^n}T, \frac{k}{2^n}T] \subset \{t : |t - s| \leq \varepsilon\}$. Moreover, by the definition of $D_\theta(B)$, there exist $t_1, t_2 \in I_{k,n}$, such that

$$|(a \cdot B_{t_1, t_2})| \geq 2^{-n\theta} D_\theta(B),$$

so t_1 or t_2 (say t_1) satisfies

$$|(a \cdot B_{s, t_1})| \geq \frac{1}{2} 2^{-n\theta} D_\theta(B).$$

According to the arbitrary choice of a, s, ε , it follows that

$$L_\theta(B) \geq \frac{1}{2} \frac{2^{-n\theta}}{\varepsilon^\theta} D_\theta(B) \geq \frac{1}{2} \left(\frac{1}{2T}\right)^\theta D_\theta(B).$$

Finally, with an application of Lemma 4.3, one arrives at inequalities

$$\begin{aligned} \hat{c}(L_\theta(B) < \tilde{\varepsilon}) &\leq \hat{c}(D_\theta(B) < 2^{1+\theta} T^\theta \tilde{\varepsilon}) \\ &\leq \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} \hat{c}\left(\inf_{|a|=1} \sup_{s, t \in I_{k,n}} |(a \cdot B_{s,t})| \leq 2^{-n\theta} 2^{1+\theta} T^\theta \tilde{\varepsilon}\right) \\ &\leq \sum_{n=1}^{\infty} 2^n A [\exp(-bT 2^{-n} \underline{\sigma}^2 2^{2n\theta} (2^{1+\theta} T^\theta \tilde{\varepsilon})^{-2}) + \exp(-bT^{-1} 2^n \bar{\sigma}^{-2} 2^{2n\theta} (2^{1+\theta} T^\theta \tilde{\varepsilon})^{-2})] \\ &\leq \sum_{n=1}^{\infty} \tilde{A} \exp(-\tilde{b}n (2^{1+\theta} T^\theta \tilde{\varepsilon})^{-2}) \\ &\leq K \exp(-l\tilde{\varepsilon}^{-2}), \end{aligned}$$

in the second last inequality of which, we apply the fact that there exist positive constants \tilde{A}, \tilde{b} , depending on $\bar{\sigma}, \underline{\sigma}$ and T , such that

$$n \ln 2 + \tilde{b}n\bar{\varepsilon}^{-2} \leq \ln \frac{\tilde{A}}{A} + b\bar{\varepsilon}^{-2} (T\underline{\sigma}^2 2^{(2\theta-1)n} \wedge \bar{\sigma}^{-2} T^{-1} 2^{(2\theta+1)n}),$$

holds uniformly over $n \geq 1, \bar{\varepsilon} \in (0, 1)$. □

Remark 4.5. According to the above proof, one could see the non-degenerateness of G is necessary. Furthermore, constants in the above bound are uniform on the bounds of $\underline{\sigma}^2 T$ and $\bar{\sigma}^{-2} T^{-1}$.

Corollary 4.6. Let B_t a d -dimensional G -Brownian motion. Then it holds that, for any $\theta > \frac{1}{2}$,

$$\overline{\lim}_{t \rightarrow s} \frac{|B_{s,t}|}{|t-s|^\theta} = \infty, \quad \forall s \in [0, T], \quad \hat{c} - q.s., \quad (4.4)$$

Proof. Indeed, one only needs to show the result in one-dimensional case. For any $\theta > \frac{1}{2}$, one can choose θ' such that $\frac{1}{2} < \theta' < \theta$. Note that $\hat{c}(L_{\theta'}(B) = 0) \leq \hat{c}(L_{\theta'}(B) < \varepsilon)$, for any $\varepsilon > 0$. According to the above proposition, $L_{\theta'}(B(\omega)) > 0$, $\hat{c} - q.s.$. By the definition of $L_{\theta'}(B(\omega))$, it holds that, for any $s \in [0, T]$,

$$\overline{\lim}_{t \rightarrow s} \frac{|B_{s,t}|}{|t-s|^\theta} \geq \overline{\lim}_{t \rightarrow s} L_{\theta'}(B) \frac{|t-s|^{\theta'}}{|t-s|^\theta} = \infty.$$

□

Example 4.1. Suppose $\bar{\sigma} > 1, \underline{\sigma} < \frac{1}{2}$ and P^1 the law of $\frac{B_-}{2}$ under P^0 , where B_- is the one-dimensional canonical process and P^0 is the Wiener measure. By the representation theorem for G -expectation, one obtains $P^0, P^1 \in \mathcal{P}$. Fix any $t \in (0, T]$, and define a measurable set

$$A = \{\langle B \rangle_t = t\}.$$

It is clear that $P^0(A) = 1, P^1(A) = 0$, so P^0, P^1 are mutually singular. Following classical methods, it is quite possible to show that B is θ -Hölder rough $P^0 - a.s.$ and $P^1 - a.s.$. However, it is nontrivial to obtain a common null set by classical stochastic analysis. Note that the capacity \hat{c} could govern infinitely many such mutually singular measures. This profit could be quite advantageous when one faces practical problems involving probability uncertainty.

Corollary 4.7. Let $\mathbf{B} = (B, \mathbb{B}), (Y, Y')(\omega) \in \mathcal{D}_{B(\omega)}^{2\alpha}([0, T], \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n))$, and $Z \in \mathcal{C}^\alpha([0, T], \mathbb{R}^n), \hat{c} - q.s.$. Furthermore, suppose (Y, Y') satisfies assumptions in Proposition 3.3. Then denote $I_t = \int_0^t Y_s dB_s + \int_0^t Z_s ds$, and $\mathcal{R} = 1 + L_\theta(B)^{-1} + \|\mathbf{B}\|_{\mathcal{C}^\alpha} + \|Y, Y'\|_{B, 2\alpha} + |Y_0| + |Y'_0| + \|Z\|_\alpha + |Z_0|$. One has the inequality

$$\|Y\|_\infty + \|Z\|_\infty \leq MR^q \|I\|_\infty^r \quad \hat{c} - q.s.,$$

for some constants M, q, r , depending only on α, θ, T .

In particular, if

$$\int_0^t Y_s dB_s + \int_0^t Z_s ds = \int_0^t Y'_s dB_s + \int_0^t Z'_s ds,$$

it holds that $Y \equiv Y', Z \equiv Z', \hat{c} - q.s.$

Proof. For any fixed α , there exists a constant $\theta \in (\frac{1}{2}, 2\alpha)$. According to Proposition 4.4, B is θ -Hölder rough, $\hat{c} - q.s.$. By applying Theorem 2.5, one could obtain the desired result. \square

Remark 4.8. According to the Norris lemma for rough paths, the above version of Norris lemma in G -framework fails to distinguish the integral with respect to $d\langle B \rangle$ and that with respect to dt , mainly because as a quadratic variation process, $\langle B \rangle$ is no longer rough any more. The distinguish of integrals with respect to $d\langle B \rangle$ and dt is done in [26] by probabilistic methods. To give a quasi-surely quantitative distinction between these two integrals, further work may need to be done in the future.

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